

BPS Saturated Vacua Interpolation along One Compact Dimension

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Abstract

A class of generalized Wess-Zumino models with distinct vacua is investigated. These models allow for BPS saturated vacua interpolation along one compact spatial dimension. The properties of these interpolations are studied.

1 Introduction

Recently, there has been a great interest in space times with D infinite and n compact spatial dimensions [1, 2]. Phenomenologically, this interest originates from the idea that the four dimensional Planck mass M_4 may be obtained from a $4+n$ dimensional fundamental mass M_{4+n} being of the order of the electroweak scale, where the n extra spatial dimensions are compact. Taking this assertion seriously, the radius R of the extra dimensions must be large ($R \leq 1$ mm for $n = 2$) [1]. In this scenario the (light) fields of the Standard Model are trapped on a defect along the compact dimensions while gravitons propagate freely. At the scale of distances set by the size of the extra dimensions the dynamics underlying the formation of the defect(s) and accompanying gravity should effectively be describable in terms of field theory rather than string theory, although a string realization of low scale gravity has been put forward in Ref. [3]. In a field theoretic context defects can appear in the form of domain walls, gauge vortices, etc. to have widths of the order of the $4+n$ dimensional Planck length [1]. In processes with high transverse momentum (comparable to the electroweak scale) it should be possible to lose energy to the bulk region and hencefore violate four-momentum conservation in our four dimensional universe. A similar picture has been set up for the case of infinite extra dimensions in Ref. [4]. There it was argued that the formation of defects in supersymmetric theories leading to the spontaneous breakdown of a part of SUSY may provide a means of trapping of the zero modes and thereby reducing the dimension of space time for these light modes. In contrast to the case of non-compact space time the emergence of parallel worlds is conceivable in the extra compact dimensions since there may be stable constellations of manifold defects as we will make explicit below.

To solidify the above scenario it is important to understand the dynamical aspects of defect generation and particle trapping. It is also necessary to explain in a detailed way how the compact dimensions can be stabilized against a possible gravitational collapse or an expansion caused by other forces [5, 6].

In the framework of supersymmetric theories it is known for a long time that the BPS nature of a vacua interpolation preserves a part of SUSY and that the very existence of it requires the central charge(s) of the SUSY algebra to be non-zero [9, 10]. Extensive investigations of BPS saturated domain walls and their junctions have been performed for example in Refs. [7, 8]. For the case of BPS vacua interpolation along one spatial dimension the $(2+1)$ dimensional dynamics of light modes due to Nambu-Goldstone fields bound to the corresponding domain walls was shown to be supersymmetric [12].

In this paper we revisit the issue of dynamical supersymmetry breaking in one compact spatial dimension. The analysis is performed for a class of generalized Wess-Zumino toy models in four dimensions with one spatial dimension compactified on a circle of radius R . These models admit BPS saturated vacua interpolation along the compact dimension.

The paper is organized as follows: In Section 2 we briefly review the peculiarities

of BPS saturation for solutions dependent on one compact, spatial dimension as they were worked out in Ref. [13]. We then introduce a class of generalized Wess-Zumino models with members labeled by an integer N . The N th model has $N + 1$ vacua which can be connected by BPS saturated solutions of the scalar sector. The properties of these vacua interpolations are investigated on the classical level, and some remarks about quantum corrections are made. Section 3 summarizes the results and comments on how the toy models considered here relate to the phenomenological notion of large extra dimensions.

2 Model and BPS solutions

In the case of (3+1)-dimensional non-compact space the vacua interpolating solutions of a supersymmetric model with a set of distinct ground states are topologically stable [13]. These solutions may or may not be BPS saturated. Specializing to a (3+1)-dimensional Wess-Zumino model with superpotential \mathcal{W} , one may seek for an interpolation of the vacua $\Phi = \Phi(z)$ ¹ satisfying the following equations of BPS saturation [12]

$$\partial_z \Phi(z) = \frac{\partial}{\partial \bar{\Phi}} \bar{\mathcal{W}} , \quad \partial_z \bar{\Phi}(z) = \frac{\partial}{\partial \Phi} \mathcal{W} , \quad (1)$$

where a possible phase factor has been absorbed in the definition of the superpotential. If such a solution exists then its tension $\tau[\Phi]$ is equal to the (1,0) central charge Z of the SUSY algebra, which has been shown long ago [9].

Since $\tau[\Phi]$ is given by the integral over a smooth, positive semi-definite function the nonvanishing of Z is a necessary condition for the existence of a BPS saturated vacua interpolation. It is given as [12]

$$Z = \int dz \frac{d}{dz} \bar{\mathcal{W}}(\Phi) , \quad (2)$$

where the integration extends over the entire dimension.

In the case of a (3+1) dimensional space with the z coordinate compactified on a circle with radius R the field Φ has to satisfy periodic boundary conditions $\Phi(0) = \Phi(L \equiv 2\pi R)$. Hence, the vanishing of Z can only be avoided if the superpotential \mathcal{W} is a multivalued function, i.e. the target manifold of field variables admits non-contractable cycles. The former possibility is unproblematic as long as the derivatives $\mathcal{W}' \equiv \frac{\partial \mathcal{W}}{\partial \Phi}$, $\mathcal{W}'' \equiv \frac{\partial^2 \mathcal{W}}{\partial \Phi^2}$, ... are single valued since it is these quantities that define the physically relevant scalar potential $V(\Phi, \bar{\Phi}) \equiv \frac{\partial \mathcal{W}}{\partial \Phi} \frac{\partial \bar{\mathcal{W}}}{\partial \bar{\Phi}}$ and the fermionic interactions [12].

Let us consider the one-parameter set of generalized Wess-Zumino models with the standard Kähler and the following superpotential

$$\mathcal{W} \equiv i \left(\log \Phi - \frac{1}{N+1} \Phi^{N+1} \right) , \quad (N = 1, 2, \dots) . \quad (3)$$

¹In the following we will use the same symbol for the chiral superfield and the scalar component field.

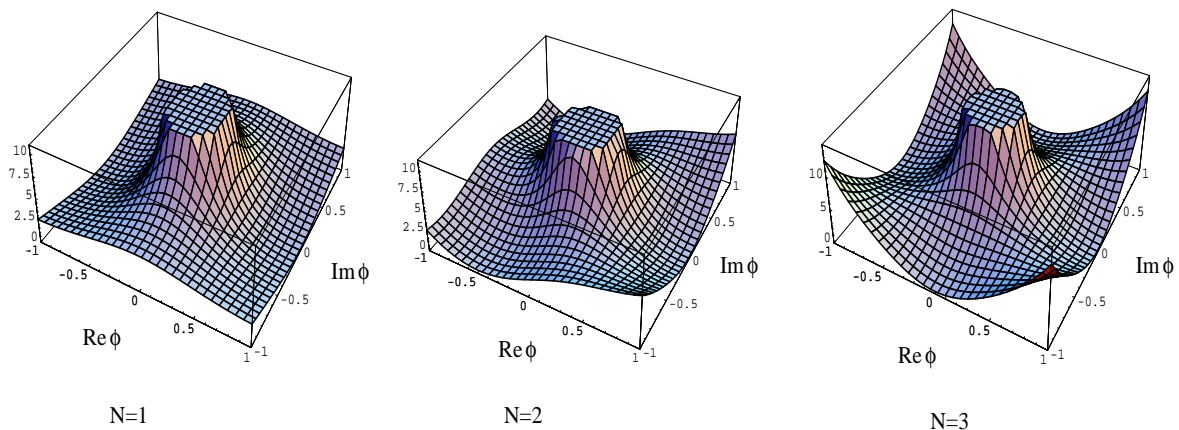


Figure 1: The scalar potential for the cases $N = 1, 2, 3$.

The vacua $\Phi_{N,k}^*$ of model N are determined by the zeros of \mathcal{W}' , and hence we have $\Phi_{N,k}^* = e^{\frac{2\pi i}{N+1}k}$, ($k = 0, \dots, N$). For $N = 1, 2, 3$ the scalar potential $V(\Phi, \bar{\Phi}) = \tilde{V}(\text{Re}\Phi, \text{Im}\Phi)$ is depicted in Fig. 1. We are interested in BPS saturated vacua interpolation $\Phi = \Phi(z)$. Using the BPS-equations (1), it is easy to see that the quantity

$$I \equiv -\text{Im } \mathcal{W} = -\log |\Phi| + 1/2 \left[(\text{Re } \Phi)^2 - (\text{Im } \Phi)^2 \right], \quad (N = 1), \quad (4)$$

is a constant of "motion" [12, 13]. From Eq. (4) it is clear that for initial values $|\Phi(0)| \ll 1$ the BPS trajectories are circles centered at the origin. Indeed, in this limit the pole in \mathcal{W}' , $\bar{\mathcal{W}}'$ is dominant (right-hand sides of (1)), and the BPS equations have the simple winding solution [13, 14]

$$\Phi_{wi} = \sqrt{L/2\pi} e^{-\frac{2\pi i}{L}z}, \quad (5)$$

which implies the relation $L = 2\pi A^2$ between the initial value $A \equiv \Phi_{wi}(0)$ (A real) and the cycle length L . There are similar solutions with winding number $n > 1$. All these solutions are uninteresting for the purpose of the present paper in the sense that (a) they do not interpolate between the vacua of the models, and (b) they have a uniform distribution of energy density.

In the case $A = 1 - \delta$ ($0 < \delta \ll 1$) Fig. 2 depicts the trajectories of the solutions to Eq. (1) for $N = 1, 2, 3$. Winding around the pole at $\Phi = 0$, these solutions (approximately) connect all vacua of the corresponding model in a clockwise sense. We argue later that for each N there is a bijection between the BPS saturated periodic solutions of a given topology n and the cycle length L , which hence can be used to label these solutions. In the periodic case the tension $\tau[\Phi_L]$ is the same for all N and L . It is given by minus the residue of the pole of \mathcal{W}' times $2\pi i$, that

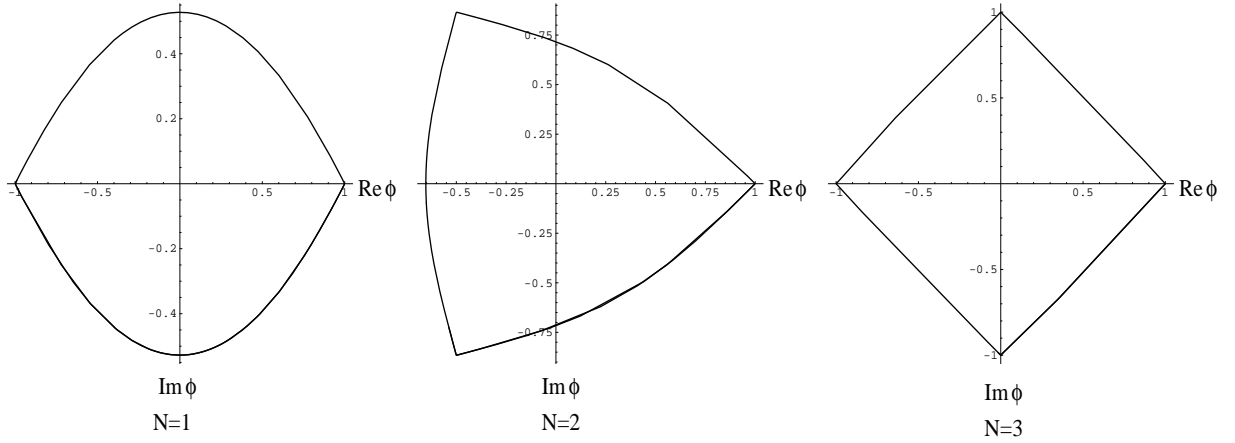


Figure 2: The BPS vacua interpolations for $N = 1, 2, 3$.

is $\tau[\Phi_L] \equiv 2\pi$. Fig. 3 shows $\text{Re } \Phi_L$ and the energy density ε_L as functions of z . In order to quantify the behavior of the solutions in the vicinity of the vacua $\Phi_{N,k}^*$ ($k = 0, \dots, N$) for the topological sector $n = 1$ we may linearize the BPS equations about these. Linearizing about $\Phi_{N,0}^* = 1$ and demanding the initial condition $\tilde{\Phi}(0) = (A - 1)$ for the deviation $\tilde{\Phi}$, we obtain the following solution

$$\tilde{\Phi}(z) = (A - 1) [\cosh((N + 1)z) + i \sinh((N + 1)z)] . \quad (6)$$

Approximating $\Phi(z)$ at $z = \frac{L}{2(N+1)}$ (first wall) as

$$1 + \tilde{\Phi}\left(\frac{L}{2(N+1)}\right) \approx \Phi\left(\frac{L}{2(N+1)}\right) \approx \cos(\pi/(N+1)) e^{-\frac{\pi i}{N+1}} , \quad (N > 1) , \quad (7)$$

and, by means of Eq. (6), comparing real parts we obtain in the limit of large L ($A \rightarrow 1$) the relation between initial value A and cycle length L

$$L = 2 \log \left[\frac{1 - \cos^2(\pi/(N+1))}{1 - A} \right] \approx \log \left[\frac{\pi^4}{N^4(1 - A)^2} \right] , \quad (N \gg 1, N^4 \ll \frac{\pi^4}{(1 - A)^2}) . \quad (8)$$

For small N Eq. (8) indicates a weak dependence of L on N in accord with the numerical findings of Fig. 3. In Fig. 4 we indicate L as a function of the initial value A for $N = 1$. The quadratic regime of the pole approximation is present up to values as high as $A = 0.8$. The above numerical calculation yields a stronger increase of L than does the pole approximation. The logarithmic regime starts at $A \approx 0.999$.

In the large N limit (with the condition $N \gg 1, N^4 \ll \pi^4/(1 - A)^2$ satisfied) the linearization of the BPS equations is justified for an entire interpolation of two

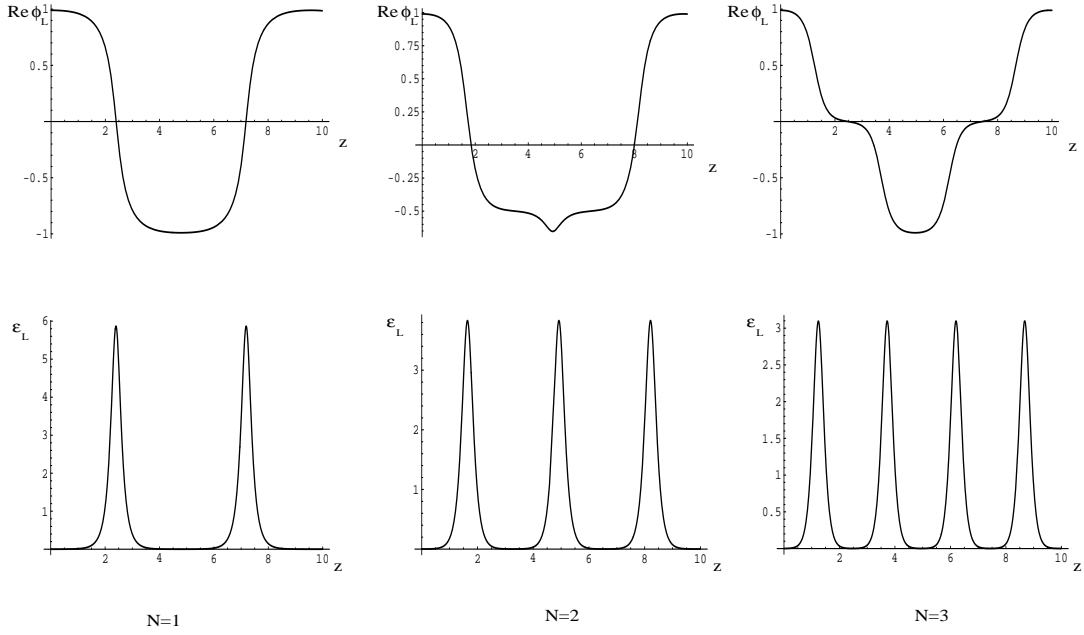


Figure 3: The real part and the energy density ε_L of the vacuum interpolating BPS solutions Φ_L for $N = 1, 2, 3$.

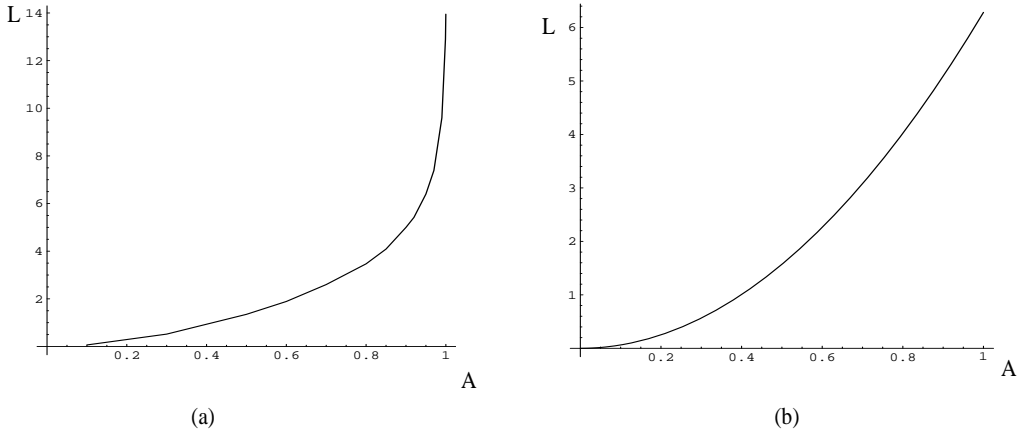


Figure 4: The cycle length L as a function of the initial value A for $N = 1$. The case (a) is the numerical result whereas (b) corresponds to the winding solutions of the pole approximation ($L = 2\pi A^2$).

adjacent vacua. Then, from Eqs. (6) and (8) we conclude that the energy density ε_L^w in the center of the wall is

$$\varepsilon_L^w \approx \frac{\pi^4}{N^2}, \quad (N \gg 1, N^4 \ll \pi^4/(1-A)^2), \quad (9)$$

that is, ε_L^w is independent of the initial value A . We know that the tension τ is independent of N and L . Defining the width Γ of each wall by

$$\tau = (N+1) \times \varepsilon_L^w \times \Gamma, \quad (10)$$

we find

$$\Gamma \approx \frac{2}{\pi^3} N, \quad (N \gg 1, N^4 \ll \pi^4/(1-A)^2). \quad (11)$$

Hence, in the above limit the sequence of domain walls loses its peak structure suggesting that there is a critical ratio $R = \pi^4/((1-A)^2 N^4) \gg 1$ above which the individual walls melt into an almost uniform energy distribution along the entire compact dimension.

What happens for $A > 1$? The solution of the linearized problem of Eq. (6) indicates that $\Phi(z)$ moves away from the origin within the first quadrant. For $N = 1$ let us assume that at some $z_0 > 0$ Φ is close to the line $e^{i\pi/4}t$, ($0 \leq t < \infty$), that is

$$\Phi(z) \approx f(z)(1+i), \quad (f \text{ positiv real for } z \geq z_0).$$

Since $\Phi(z)$ initially moves away from the origin we furthermore can assume that f is considerably larger than unity. Omitting then the pole term in the right-hand side of the BPS equations, we deduce that $\Phi(z) = f(z)(1+i)$ is an approximate solution for $z \geq z_0$ with $f(z) = e^z$. This blow-up behavior is observed numerically also for $N = 2, 3$. The corresponding solutions are not periodic, and therefore they are not relevant to the case of a compactified dimension. For $N = 1$ (and presumably also for $N > 1$) we thus have found that the cycle length L of the periodic (non-constant) BPS solutions as a function of the initial value A is a bijection $L : (0, 1) \rightarrow (0, \infty)$. The set of periodic BPS solutions at given L is therefore uniquely indexed by the winding number $n = 1, 2, \dots$ ² Rescaling the Kähler metric, we therefore conclude that BPS saturation is possible in all regimes of coupling strength (see Ref. [13] for details). We can view the BPS interpolation $\Phi_L(z)$ at cycle length L as the classical ground state of the model. It is known for a long time that the tension of this configuration does not receive any perturbative corrections [12]. Since the mass of

²There is also a bijection between the periodic BPS solutions and the initial value $A \in (0, 1)$: Since the right-hand sides of the BPS equations (1) are smooth in the entire complex plane except for the point zero the solutions are uniquely determined by the initial value. It remains to be shown that *each* periodic solution is associated with one value of $A \in (0, 1)$. Assume this was not the case, that is, there is a periodic solution $\Phi_B(z)$ with initial value $B \notin \{\text{trajectories labeled by } A\}$. At some $z_R > 0$ Φ_B must cross the real axis with $0 < \Phi_B(z_R) < 1$ for otherwise it will be not periodic as we showed above. Setting $A \equiv \Phi_B(z_R)$, we obtain a contradiction to the assumption that B coincides with no point of the trajectories labeled by A .

the BPS interpolation is infinite, nonperturbative effects like instanton-tunneling can not change the BPS tension either.

Regarding the superpotential of Eq. (3) to define a generalized Landau-Ginzburg model [13] with extended $\mathcal{N} = 2$ SUSY in (1+1) dimensions (spatial dimension compact) [13], the infinite-mass argument would no longer hold. A priori, there could be instanton mediated tunneling to change the energy of the state and make it non-BPS [13, 15]. However, since we have a bijection between cycle length and periodic solution there is only one BPS state with given winding number n at a given radius R . Hence, non-perturbative quantum tunneling within one topological sector is impossible.

3 Summary

The existence of BPS saturated vacua interpolations along one compact spatial dimension has been exemplarily demonstrated for a class of generalized Wess-Zumino models in (3+1) dimensions. The non-vanishing of the $(1, 0)$ central charge Z of the SUSY algebra, necessary for the existence of BPS saturated solutions, is achieved by a superpotential with a branch cut. The solutions of the BPS equations are either periodic (tension= 2π) or blow-up (infinite tension). For the case $N = 1$ we showed explicitly that there is a bijection between the set of periodic solutions and the cycle length $0 < L < \infty$. In the limit of large L and N the width Γ of the walls scales as N . On the other hand, for fixed initial value $0 < A < 1$ the length of the cycle scales logarithmically with N suggesting that there are critical combinations of the quantities N and A for which the individual walls start to melt into a uniform distribution of energy throughout the entire compact dimension.

Viewing the above class of models as $\mathcal{N} = 2$ generalized Landau-Ginzburg models in (1+1) dimensions [13], the energy of the classical BPS state of cycle length L does not get affected by non-perturbative instanton tunneling for $N = 1$ (presumably this is also true for $N > 1$). Perturbative corrections in general change the shape of the BPS interpolation but leave its energy (tension) intact [12].

The class of $(3 + 1)$ dimensional models that were investigated in this paper are toy models in the following sense: Since four is the maximal dimension at which one can build a non-trivial, supersymmetric theory of scalars and fermions Wess-Zumino models cannot make direct contact with the phenomenology of extra dimensions as put forward in Refs. [1, 2, 3]. However, they do provide a simple framework for the visualization of compactification scenarios. The possibility of the dynamical generation of a stable "brane-lattice", which may cause a gravitational stabilization of the extra dimension [6], is transparent in the setting of a Wess-Zumino model. Concerning the question of SUSY breaking in the universe of light particles trapped on a spontaneously generated defect, one may introduce small SUSY breaking terms into the scalar potential, which allow for a perturbative treatment and periodicity of the perturbed solution. Alternatively, one could start with a non-supersymmetric

model allowing for BPS saturated domain walls. The reason for the existence of light, localized fermions is then solely due to generalized index theorems [11] in contrast to the case of a spontaneous, partial breakdown of SUSY, where the localized fermions primarily are Goldstinos [12].

Acknowledgements

The author likes to thank M. Shifman for reading the manuscript and suggesting various improvements. Stimulating and useful conversations with T. ter Veldhuis are gratefully acknowledged. This work was funded by a postdoctoral fellowship of Deutscher Akademischer Austauschdienst (DAAD).

References

- [1] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Lett.**B57**, (1998) 263
- [2] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, Phys. Rev.**D59**, (1999) 086004
- [3] I. Antoniadis and N. Arkani-Hamed and S. Dimopoulos and G. Dvali, Phys. Lett.**B436**, (1998) 257, hep-ph/9804398
- [4] G. Dvali and M. Shifman, Nucl. Phys.**B504**, (1997) 127
- [5] R. Sundrum, Phys. Rev.**D59**, (1999) 085010
- [6] N. Arkani-Hamed, S. Dimopoulos, and G. Dvali, (1998) hep-th/9809124
- [7] D. Bazeia and F. A. Brito, Phys. Rev. Lett.**84**, (2000) 1094, hep-th/9908090
- [8] D. Bazeia and H. Boschi-Filho and F. A. Brito, JHEP **04**, (1999) 028, hep-th/9811084
- [9] E. Witten and D. Olive, Phys. Lett.**B78**, (1978) 97
- [10] A. Gorsky and M. Shifman, Phys. Rev.**D61**, (2000) 08500
- [11] R. Jackiw and C. Rebbi, Phys. Rev.**D13**, (1976) 3399
- [12] B. Chibisov and M. Shifman, Phys. Rev.**D56**, (1997) 7990; Erratum **D58**, (1998) 109901
- [13] X. Hou, A. Losev, and M. Shifman, Phys. Rev.**D61**, (2000) 085005
- [14] G. Dvali and M. Shifman, Phys. Lett.**B454**, (1999) 277
- [15] E. Witten, J. Diff. Geom.**17**, (1982) 661